Philosophy of quantum mechanics

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These notes are intended as a didactic material for the master and PhD students of the course *Epistemology and Philosophy of Science* organized by the Department of Philosophy at the University of Lisbon in the academic year 2022/23. They cover all the topics that will be presented during the lessons: (i) the formalism and the standard interpretation of quantum mechanics; (ii) the measurement problem; (iii) the different solutions to the measurement problem, namely the de Broglie–Bohm, the spontaneous collapse (GRW) and the Everett theory; (iv) the meaning of the wave function; (v) decoherence theory and the quantum-to-classical transition. All these topics are presented having in mind students with no previous background in mathematics or physics. The choice to present the formalism of quantum mechanics before the philosophical discussion comes from my personal belief that these two aspects cannot be really separated in any serious attempt to discuss the philosophy of quantum mechanics. Furthermore, given the introductory nature of these notes, they can serve as a first step for all the students aiming to do research in the sometimes exoteric but fascinating field of philosophy of physics.

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1 Introduction

1.1 What is quantum mechanics?

Quantum mechanics is the theory of physics that describes objects on very small scales. There is no precise cut off between quantum mechanics and classical mechanics,[1] but a typical scale in which classical mechanics fails to describe objects is the molecular scale. Macroscopic objects such as tables, chairs, cats, human beings are correctly described by classical mechanics. Microscopic objects such as electrons, protons, nuclei, molecules are correctly described by quantum mechanics. There is however no precise boundary between the definitely macroscopic region (tables, chairs) and definitely microscopic region (electrons, nuclei); the region "in-between" these two is generally called the mesoscopic region, and corresponds typically to objects with the size of large molecules. Roughly,

[1] As we will see in the chapter dedicated to decoherence, classical mechanics emerges from quantum mechanics in the macroscopic regime.

we can say that objects bigger than 10^{-6}m are well described by classical mechanics, whereas objects smaller than 10^{-9} are well described by quantum mechanics. The intermediate range $10^{-9} - 10^{-6}$m is the mesoscopic scale, where quantum effects and effective classical descriptions are generally combined together.[2]

microscopic (QM)	mesoscopic	macroscopic (CM)
$< 10^{-9}$m	$10^{-9} - 10^{-6}$m	$>10^{-6}$m

So, quantum mechanics is the theory that describes objects at the microscopic scale, from elementary particles to individual molecules,[3] but *what is quantum mechanics about*? That is: how does quantum mechanics describe molecules, atomic orbitals and elementary particles? What does quantum mechanics tell us about the invisible microscopic world? Atoms and molecules are the basic constituents of all material objects: understanding the description of those objects given by quantum mechanics means therefore understanding (what we know nowadays of) the nature of matter.

1.2 The structure of a physical theory

The structure of a physical theory can be summarized by the following three elements:

The system state: we call *systems* all the objects that are represented by the theory. In classical mechanics, systems will be typically tables, chairs, planets; in quantum mechanics, systems will be atoms, molecules, elementary particles. The *state* of a system is the mathematical representation of the system within the theory. any physical theory thus represent the behavior of the system through the state. The state can be represented for example by a single variable x, a set of variables (x, y, z), a function $f(t)$ or even more abstract mathematical objects, such as a vector or a tensor.

The state space: the mathematical space in which the state is defined. Many times, the state space is just a mathematical space and does not correspond to the physical

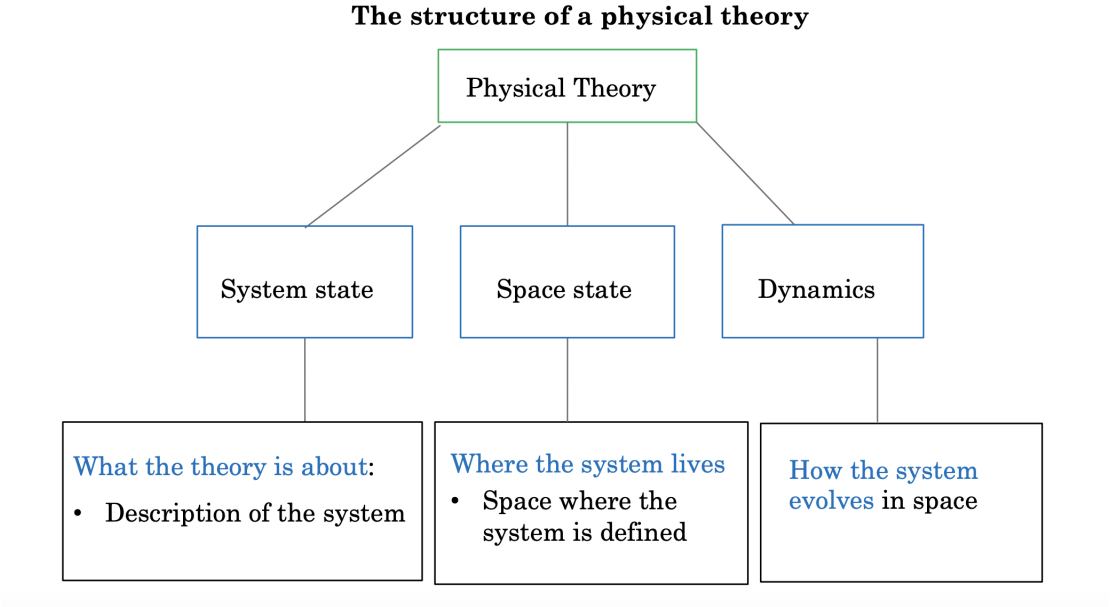
[2] The mesoscopic regime is of great interest e.g. for the material science and condensed matter. Most of nanotechnology and, in general, the attempts to exploit quantum effects in new materials and devices happens at this scale.

[3] For an accurate description of the behavior of elementary particles is often necessary to go beyond non-relativistic quantum mechanics, which is the object of the course, and to describe the elementary particles with the quantum field theory, an extension of quantum mechanics in the relativistic regime. This is for example the theory used at CERN to describe the collisions between particles and study the effects of such collisions. Quantum field theory is therefore the mathematical framework of particle physics, and the theory–or better the collection of theories–describing particle physics is called the standard model. However, non-relativistic quantum mechanics provides the very basic framework of such theories: all the weird physical behavior of elementary particles already emerges in non-relativistic quantum mechanics. And this is the reason why many philosophical discussions have quantum mechanics as a framework. Furthermore, most of the conceptual problems that arise in quantum mechanics–the meaning of the wave function, non-locality, the collapse of the wave function–are not alleviated (on the contrary, they quickly become worse and much less manageable) by the framework of quantum field theory.

space where the system really exists. For example, in quantum mechanics the state space is the system configuration space, a space with $3N$-dimensions, where N is the number of particles that compose the system. Even thought there are authors arguing that this is indeed a legitimate physical space for quantum systems, the community overall agrees that this is best regarded as a fictitious mathematical space and should not be reified as a physical space.

The dynamics: how the system evolves in time. Mathematically, this is represented by the evolution of the state (description of the system) in the state space (where the system is mathematically defined). The dynamics of a theory may be intuitive, as in Newtonian mechanics, or more abstract, as in quantum mechanics. In the former, the dynamics of a system is given by Newton's second law $F = ma$, in the latter the dynamics is given by the Schrödinger equation $i\hbar\dot{\psi} = \hat{H}\psi$, which will be introduced in section [2] and analyzed in more detail in section [3] of these notes.

We can thus summarize the structure of a physical theory with the following schema:



2 What is quantum mechanics about?

What is quantum mechanics about? In the previous section, we have seen that the theory describes microscopic objects. But now we are interested to address a different question, that is: what sort of objects are described by the theory:

- are they particles?
- are they waves?

- are they both?
- are they a new kind of entities?

Interestingly enough, this is an open question in the philosophy of quantum mechanics, and all the answers quoted above have been strongly defended by different authors.[4] However, leaving aside for the moment possible ontological problems, from the physical point of view this question has a simple answer, also in quantum mechanics. In fact, "what the theory is about" is mathematically represented by the *state* of the system described in the theory. And the state is generally indicated by the equations of motion that characterize the theory. It could be useful, in this regard, to consider first two examples of classical theories (Newton's theory and Maxwell's electromagnetism), and then coming to the analysis in quantum mechanics.

2.1 Example 1: Newtonian mechanics

What Newton's theory is about? Let us look, for example, at the equations of motion of the theory. The equations of motion, or dynamical equations, are the equations that describe the dynamics of the theory, i.e. how the objects described by the theory evolve in time. In Newtonian mechanics, the principal equation of motion is Newton's second law of dynamics:

$$F = ma = m\ddot{x} \tag{1}$$

From eq. (1), we see that the theory is about massive objects (m) having well-defined positions in space (x) and accelerating under the action of the force F. The force can be the gravitational force (which is the only genuine/ontological force in Newton's theory) or some effective force such as the elastic force or friction (these originate at the microscopic level but have a concrete effect on the motion of the macroscopic body. So they are effectively accounted in Newton's theory). The variable x is quite flexible: the theory can be extended from point-particles to extended bodies. In the case of an extended body, the x generally represents its center of mass.
From the dynamical point of view, classical particles (and bodies) move along well-defined trajectories in three-dimensional space. The continuity of the trajectory can be derived from the fact that Newton's second law of dynamics is a differential equation that has a unique solution at each instant of time if the initial conditions are known. The initial conditions in Newton's theory are the initial position and initial velocity of the particle. If at the initial time ($t = 0$) we know the position $x_0 = x(t = 0)$ and velocity $v_0 = v(t = 0)$ of the particle, and is also known the force $F_0 = F(t = 0)$ acting on the particle, then we have a unique solution for the position and the velocity of the system at all later times. In short, the dynamics of the particle is deterministic: having the exact knowledge of the relevant variables (positions, velocities, force) at a given instant of time, we can predict the further evolution of the system or retrodict its past evolution.

[4] This debate and the different answers will be abalyzed in the chapter dedicated to the meaning of the wave function.

This is what the theory tells us about the world: there are particles, those particles move along continuous trajectories in space and the motion of the particles is caused by forces, in particular by the gravitational force.[5]

2.2 Example 2: Classical electromagnetism

What classical electromagnetism is about? The fundamental equations of the theory are in this case the Maxwell's equations:

$\nabla \cdot E = \frac{\rho}{\epsilon_0}$	$\nabla \cdot B = 0$
$\nabla \times E = -\frac{\partial B}{\partial t}$	$\nabla \times B = \mu_o J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$

These equations describe two physical entities, the electric field $E(x,t)$ and the magnetic field $B(x,t)$ that are originated from an actual distribution of electric charge represented by ρ (density charge). The physical source of the field is manifest in the first Maxwell's equation of the electric field, whereas in the case of the magnetic field this is "hidden" into the density current $J = \rho v$. From Maxwell's equations, we see that classical electromagnetism is about two physical fields: $E(x,t)$ and $B(x,t)$, respectively the electric and magnetic field. These two fields are coupled together: the configuration of one field influences the behavior of the other field and *viceversa*, This is regarded as a sign that there is in fact only one physical entity: electromagnetic field, a fusion (coupling) of the electric and magnetic field. So, also in this case, from the central dynamical equations we know what classical electromagnetism is about: it is about a vector physical field, the electromagnetic field, having a precise configuration in space, evolving in time according to Maxwell's equations. From the ontological point of view, classical electromagnetism is not only an addition but also a generalization with respect to Newtonian mechanics, as it introduces the existence of a more general entity, the physical field, in the fundamental structure of the world. A field is an abstract physical entity that is mathematically well-defined in a limited region of space. The specification of a field follows from the specification of the field-values at each point of the region in which the field is well-defined. The field values can be numbers (scalar field) or vectors (vector field). Electric and magnetic fields are *vector fields*, whereas, for example, the wave function in quantum mechanics (which mathematically looks like a field) is a *scalar field* defined on configuration space (see next section).

Note that the electric and magnetic fields cannot be measured or observed directly, but in indirect way through the motion of charged particles. If we insert a charged particle in a region over which is defined an electric and magnetic field, the charged particle will move under the force generated by the fields. In particular, a particle with charge q and velocity $\vec{v}$ will be acted upon by the *Lorentz force*:

[5] This is the only genuine force acting in classical Newton's theory. Other types of force, such as the elastic force or friction, are instead effective force ultimately reducible to electromagnetic forces between microscopic elementary components of the macroscopic classical system. Those forces however are taken into account in Newton's theory when they have a non-negligible impact on the dynamical behavior of the system.

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{2}$$

We may say that the existence of the electric and magnetic fields is observed or measured indirectly from the motion of the charged particle.

2.3 Example 3: Quantum mechanics

What is quantum mechanics about? In analogy with the examples above, let us look at the central equation of motion. In this case, the situation is even simpler than Newtonian mechanics and classical electromagnetism, as in quantum mechanics there is only one equation of motion, the Schrödinger equation:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \hat{H}\psi(x,t) \tag{3}$$

where $i = \sqrt{-1}$ is the imaginary unit, $\hbar = \frac{h}{2\pi}$ is the Planck constant h[6] divided by 2π, the symbol ∂ indicates a partial derivative (a derivative of the object in the numerator only respect to the variable in the denominator), $\hat{H}$ is the Hamiltonian operator, which represents the total energy of the system, the function $\Psi(x,t)$ is the wave function, which represents the state of the quantum system. We are now in the position to answer the question: *quantum mechanics is a theory about the wave function.* Systems in quantum mechanics are mathematically represented by the wave function, which is the solution of the Schrödinger equation as the electric and magnetic fields are the solutions of Maxwell's equations. The evolution of the wave function is fixed by the Schrödinger equation and, since this is a differential equation, the wave function evolves deterministically: given the specific functional form of the wave function at the initial time ($t = 0$) and the necessary boundary conditions, we can predict with certainty the further evolution of the wave function at all later times. The probabilistic character of quantum mechanics does not arise thus from the Schrödinger equation. As we will see in the next section, quantum probabilities emerge in the measurement interactions: every time we want to have information on some physical property of the system, this information is granted probabilistically.[7] Elementary particles, such as electrons and neutrons[8], atomic orbitals,

[6] (the Planck constant h has a very small value, it is a number that characterizes quantum systems as it indicates the minimum separation between different energy levels of a system.The fact that the energy levels of a quantum systems are discrete (with a minimum distance between them proportional to the Planck constant) was historically one of the first discoveries and success of quantum theory. The term "quantum" follows indeed from this process of discreteness or "quantization" between different energy levels related to the spectrum of the atomic orbitals.

[7] There is an exception: if the system's state is an eigenstate of the operator representing the measured observable, then the measurement outcome can be predicted with certainty. This statement may look exoteric at his stage, but it will be clear after presented the formalism of the theory.

[8] With "elementary particles" I just refer here to the basic constituents of an atom. Strictly speaking, neutrons and protons are not elementary particles, as they are composed of quarks. Quarks, however, are not described by (non-relativistic) quantum mechanics but by a relativistic extension of quantum mechanics called *quantum field theory.* In particular, the physics of quarks is described by quantum cronodynamics (QCD) whereas quantum electrodynamics (QED) describes the interaction between light (photons) and matter. QED and QCD together form the Standard Model of particle physics.

molecules are described by the wave function. But the problem is just shifted:

- what is the wave function?

This is still an open question. At the moment, many proposals have been advanced and defended by different authors, but no consensus has been reached on one specific proposal.[9] At a first glance, the wave function is quite similar to a field. As for the electric and magnetic fields, the wave function is the solution of a dynamical equation and describes a physical entity that is spread in space. However, differently from the electric and magnetic fields, the wave function has some peculiar features that make it difficult to interpret as a real physical field. Specifically:

1. **The wave function is a complex scalar field:** the "field-values" of the wave function are complex numbers. Complex numbers are not measurable (they cannot be even represented on a line but only on a plane) and so generally are not associated to physical entities. For example, in classical electromagnetism, phasors are complex numbers: they represent indeed the coupling between the amplitude and the phase of the electromagnetic wave, which are real-valued measurable quantities and correspond to real features of the electromagnetic wave.

2. **The wave function is defined in configuration space:** which is different (for systems composed of more than one particle) from the ordinary three-dimensional space. The system's configuration space has dimensions $3N$, where N is the number of particles composing the system. For example, the wave function of a 2-particle system is mathematically defined in $3 \times 2 = 6$ dimensions, the wave function of a 3-particle system in $3 \times 3 = 9$ dimensions, the wave function of a cat in $3 \times 10^{23} = 30^{23}$ dimensions.

Finally, we can represent the system state, the space state and the dynamics of the physical theories introduced so far as follows:

Physical theory	System state	Space state	Dynamics
Newtonian mechanics	(x, v)	3D space	$\vec{F} = m\vec{a} = m\ddot{\vec{x}}$
Classical electromagnetism	$\vec{E}(x,t)$, $\vec{B}(x,t)$	3D space	Maxwell's equations
Quantum mechanics I	$\|\psi\rangle$	Hilbert space	$i\hbar\frac{d\|\psi\rangle}{dt} = \hat{H}\|\psi\rangle$
Quantum mechanics II	$\psi(x,t)$	Configuration space	$i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$

[9] There is also one factor that makes the situation even more complicated: many proposals acquire a precise meaning only in a specific interpretation of quantum mechanics. This makes the spectrum of possible options wider or smaller according to the interpretations that one is ready to adopt. In short, these two problems, the meaning of the wave function and the interpretation of quantum mechanics are deeply interrelated.

2.4 The standard view: the statistical interpretation of the wave function

We call *standard interpretation of quantum mechanics* the interpretation presented in quantum mechanics textbooks and adopted by the majority of the physics community.[10] In the standard interpretation, the wave function is not a physical entity by itself. In this view, the wave function is best regarded as a computational tool to describe the system's state and its future evolution. A physical meaning is associated instead to the absolute square of the wave function: $|\Psi(x,t)|^2$, which is interpreted as a *probability density*. Note that while the wave function $\psi(x,t)$ is a complex-valued function, the absolute square of the wave function $|\psi|^2$ is a real-valued function. In general, given a complex number or a complex function z, its absolute square is, respectively, a real-valued number or a real-valued function. Let's see why: suppose that $z = a + ib$ is a complex umber and $z^* = a - ib$ is its complex coniugate, the absolute square of z: $|z|^2$ is computed as follows:

$$|z|^2 = z^* z = (a - ib)(a + ib) = a^2 + iab - aib - i^2 b = a^2 - i^2 b = a^2 + b^2$$

In the last passage we have used the property of the imaginary unit: $i^2 = -1$. Since a and b are real numbers, the quantity $|z|^2 = a^2 + b^2$ is a positive real-valued quantity and thus can be interpreted as a probability density.
In the standard interpretation of quantum mechanics, the absolute square of the wave function $|\Psi(x,t)|^2$ is interpreted as *the probability density to find the particles composing the system on a certain region of space if we perform a position-measurement on the system*. For example, given a 1-particle system with wave function ψ well-defined on the region L, the quantity $|\psi|^2$ corresponds to the probability density to find the particle in one point of the region L in a position-measurement. Note that, according to the standard interpretation, $|\psi|^2$ cannot be interpreted as the probability density that the particle *is* in a certain region, but only that the particle will be found on that region in a position-measurement. This statement is very important: it marks the difference between an epistemic probability and an intrinsic or ontological probability:

- *Epistemic probability:* the particle is actually in a specific location independently and before the measurement. However, I do not know the exact location of the particle and this is eventually "revealed" performing the measurement of position. [Classical statistical mechanics]
- *Ontological probability:* the particle does not have a precise location before the measurement. When we perform a position-measurement of the system, the particle suddenly becomes perfectly well-localized in one point in space. [Quantum mechanics: the "localization" process is the collapse of the wave function.[11]]

[10] Historical note: the standard interpretation adopted nowadays is a combination between the *Copenaghen interpretation* proposed by Bohr, in which classical measurement apparata are the only devices through which we can observe the behavior of a quantum system, and the *von Neumann-Dirac formalism*, in which measurement apparata are not classical but quantum systems (they are represented by state vectors or wave functions as any other quantum system), but they still play a special role in the theory as they are responsible for the collapse of the wave function.

[11] The collapse of the wave function is one of the most problematic aspects of the standard interpretation of quantum mechanics. This will be analyzed in more detail in the next sections.

The integral of the probability density in a given region of space (for example, the unidimensional region L) corresponds to the probability to find the particle on that region in a position-measurement of the system:

$$P = \int_L |\psi|^2 dx \tag{4}$$

This is known as the *statistical interpretation of the wave function*. Historically, it was proposed by Max Born and quite unanimously adopted by the founding father of quantum mechanics. However, even if the Born's statistical interpretation was adopted in practice by the community, not all physicists were completely satisfied by it. For example, Schrödinger was never convinced by the probabilistic interpretation and made a serious attempt to interpret the wave function as a material field composing the objects, in particular as a charge-density field distributed in space (see e.g. Callender (2020)). In his interpretation, objects with a precise position in space, such as macroscopic classical objects, are described by wave packets, i.e. wave functions well-localized in a small region of space. Nevertheless, this interpretation could not explain e.g. why the wave packets remain stable in time, as wave functions generally tend to spread out. Compared to Schrödinger's charge-density interpretation, Born's statistical interpretation seemed less artificial and, most importantly, it was an effective solution to the two main issues listed above: the fact that the wave function is a complex-valued function and that is defined on configuration space. Since, in the statistical interpretation, the wave function was not interpreted as a physical field, these two aspects were regarded as unproblematic mathematical aspects and not as physical features of the theory.[12] At the same time, the interpretation in terms of probability density is all we need in quantum mechanics to predict the measurement outcomes in any experiment. The statistical interpretation was then regarded as the most natural and less problematic interpretation to adopt. And still is, by the vast majority of working physicists. This interpretation shows its limits however when we want to go beyond the predictive power of quantum mechanics and ask for an ontological story about the processes described by quantum mechanics. As described above, the state of an electron, an atomic orbital, a molecule is represented by the wave function. We can certainly interpret $|\psi|^2$ in terms of probability density,, but this does not alleviate the problem to characterize *what an atom is, what a molecule is, what matter is basically made of.* Furthermore, the standard interpretation of quantum mechanics runs into problems if we want to describe systems independently of the measurement context. The most serious of these problems is the measurement problem. Nevertheless, before

[12] Physicists were already used to introduce abstract mathematical entities in physics: in classical mechanics, the Lagrangian function is defined on configuration space, and the Hamiltonian function is defined in phase-space. Both functions were used and are still used effectively to describe the behavior of a classical system. The analogy with the wave function is very strong: the wave function–as the Lagrangian and the Hamiltonian–is not a physical entity but a function through which we can study the behavior of a quantum system. In this case, the behavior of a quantum system is the probability for the measurement outcomes of the measured physical property or (when the physical property is position) the probability that the particle will be localized in a certain region of space. Born's statistical interpretation permitted to solve the problem of the interpretation of the wave function but at a high cost: standard quantum mechanics became essentially a theory of measurement.

turning our attention to the measurement problem, I will introduce in the next chapter the basic formalism of standard quantum mechanics. This will serve as a starting point to discuss the measurement problem and the non-standard interpretations of quantum mechanics: the de Broglie–Bohm theory, the GRW theory, the Everett–Many Worlds interpretation.

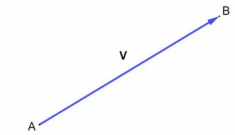
3 The structure of quantum mechanics

In this section, I will present the basic formalism of quantum mechanics. This will be divided in the following steps:

- The representation of the state: the state-vector and the wave funzion
- Introducing physical properties: observables and operators
- The dynamics of the system: the Schrödinger equation
- The theory of measurement: the collapse postulate and the Born's rule
- Empirical detection of quantum interference: the double slit experiment
- Heisenberg uncertainty principle: formalism and philosophical considerations

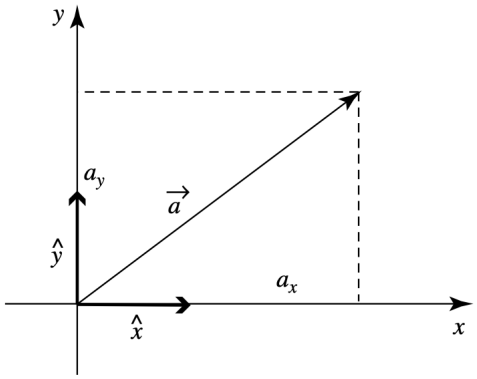
3.1 Vectors

A vector $\vec{v}$ is a mathematical object defined by a length, also called magnitude, and a direction. It can be represented geometrically by a directed segment, whose length is the magnitude of the vector and the arrow is the direction of the vector:



The mathematical space in which vectors are defined is called *vector space*. In mathematics, a space represents a set of entities and specific rules for those entities. A vector space is a space whose elements are vectors and that defines mathematical operations between vectors. All vector spaces have vectors as elements, but different vector spaces define different operations between vectors.[13] In order to describe interesting properties of vectors, it is useful to assign a reference system. For example, we describe a vector $\vec{a}$ in the Cartesian xy plane, i.e. a reference system with orthogonal basis axes: $\hat{x}$ and $\hat{y}$:

[13] Quantum mechanics is defined on Hilbert space, a complex vector space with a scalar product (also called inner product) and the definition of magnitude.



We see that the vector $\vec{a}$ can be represented as a linear combination (weighted sum) of the unitary basis vectors $\hat{x}$ and $\hat{y}$. The two components of the sum are the projection of $\vec{a}$ onto, respectively, the $\hat{x}$ axis (a_x) and the $\hat{y}$ axis (a_y):

$$\vec{a} = a_x\hat{x} + a_y\hat{y} \tag{5}$$

The vectors $\hat{x}$ and $\hat{y}$, represented with a hat instead of a arrow, are vectors of unit length = 1. Such vectors are called normalized vectors, and an orthogonal basis of unit vectors is called orthonormal basis. In quantum mechanics, every state is represented by a unit complex vector in Hilbert space, and every basis we chose to represent a state is generally an orthonormal basis of the system's Hilbert space. The linear combination that we see in eq. 5 is the origin of quantum superpositions: as a vector can be represented by a linear combination of basis vectors, a quantum state can be represented by a linear combination of different states. The superposition principle, one of the most striking properties of quantum mechanics, is rooted in the Hilbert space formalism and the vector representation. As we will see, the superposition principle will be maintained in the wave function representation: since the Schrödinger equation is linear, given two solutions:

$$\psi_1 \quad \psi_2$$

its linear combination is also a solution:

$$\Psi = c_1\psi_1 + c_2\psi_2$$

Since the Schrödinger equation is linear, the superposition principle holds not only for state vectors but also for the position-space representation of the state vector, i.e. the wave function. Therefore the superposition principle of quantum mechanics–the fact that a system can be in many well-defined states at the same time–will be characteristic only in the wave function representation.

3.2 The Dirac formalism

The state vector in quantum mechanics is generally represented in the Dirac formalism. In the Dirac formalism, a vector $\vec{v}$ is written as $|v\rangle$ (the symbol $|\cdot\rangle$ reads as "ket"). The

vector space of quantum mechanics is the Hilbert space, an infinite-dimension complex vector space with a norm (a way to compute magnitudes of vectors). Since it is a complex vector space, given a vector $\vec{v}$ its complex coniugate $\vec{v^*}$ will be also an element of the space. In the Dirac formalism, the complex coniugate of $|v\rangle$ is represented by $\langle v|$ (the symbol $\langle\cdot|$ reads as "bra"). This is why the Dirac formalism is also known as the "bra-ket" or "bracket" formalism. The scalar product between two vectors, for example: $\vec{v}\cdot\vec{w} = |v||w|cos\alpha$ is represented by the *braket* $\langle v|w\rangle$. So, the bracket $\langle v|w\rangle$ has the usual geometrical meaning of the scalar product: it is the projection of the vector $|v\rangle$ on the vector $|w\rangle$ multiplied by the magnitude of vector $|w\rangle$: $|w|$. Given $|v\rangle$, the bracket $\langle v!v\rangle$ is the projection of the vector $|v\rangle$ on itself, and the norm of the vector is given by: $\sqrt{\langle v|v\rangle}$. Note that the square root is necessary in the definition of the norm because the bracket multiply the magnitude for itself. For example, suppose the magnitude of $|v\rangle$ is $|v| = 2$. The absolute value of the bracket is $|\langle v|v\rangle = |v^*||v|cos\alpha = 2x2 = 4$, which is the square of the absolute value $|v| = 2$. This correct absolute value is captured instead by the definition of the norm: $|v| = \sqrt{\langle v|v\rangle}$. Here is a table to "translate" the ordinary formalism of vectors in the Dirac bracket formalism of quantum mechanics:

Vector formalism	Dirac formalism
$\vec{v}$	$\|v\rangle$
$\vec{v^*}$	$\langle v\|$
$\vec{v}\cdot\vec{w}$	$\langle v\|w\rangle$
$\|v\|$	$\sqrt{\langle v\|v\rangle}$

For example, the linear combination of vectors (5) in the Dirac bracket notation becomes:

$$|a\rangle = a_x\,|x\rangle + a_y\,|y\rangle \tag{6}$$

where:

- $|x\rangle$ and $|y\rangle$ are called the *basis vectors*, as they are the vector that compose (the fundamental components of) the vector $|a\rangle$
- a_x and a_y are called the *expansion coefficients*, they can be computed as follows: $a_x = \langle x|a\rangle$ and $\langle y!a\rangle$ and (in QM) they are generally complex numbers.

In quantum mechanics, a state (any possible state for the system) is mathematically represented by a vector in the Hilbert space. since a vector is naturally represented as a linear combination of basis vectors, any quantum state is generally represented as a linear combination of the basis vectors:

$$|K\rangle = c_1\,|k_1\rangle + c_2\,|k_2\rangle \tag{7}$$

where $|k_1\rangle$ and $|k_2\rangle$ are the basis vectors and c_1 and c_2 are the expansion coefficients. Since the basis vectors $|k_1\rangle$ and $|k_2\rangle$ are also vectors in Hilbert space, and so they are also legitimate quantum states, we can say that the state $|K\rangle$ is a linear combination or or a *superposition* of the states $|k_1\rangle$ and $|\,|k_2\rangle$. Therefore, in quantum mechanics, a

state is generally as a superposition of (basis) states.The basis states are arbitrary but we generally choose the states that are useful or relevant for the description of the system. In many cases, the basis states will be the states that represent well-defined values for the physical property that we want to study. Suppose, for example, that we want to study the energy of the system S with state $|S\rangle$. In this case, it would be useful to expand the state in the energy eigenstates $||E_i\rangle$:

$$|S\rangle = c_1 |E_1\rangle + c_2 |E_2\rangle + \dots \ c_N |E_N\rangle = \sum_{i=1}^{N} c_i |E_i\rangle \tag{8}$$

the states $|E_i\rangle$ are states for which the energy of the system is well-defined. In general, a state for which the system has a well-defined value for a given physical property is called an *eigenstate* of that physical property. So, in this case, the states $|E_i\rangle$ are the eigenstates of the energy operator (a physical property is always represented mathematically by an operator, in particular by an Hermitian operator) or simply energy eigenstates.[14]

To resume: in quantum mechanics a *state* is mathematically represented by a vector in Hilbert space and can be represented as a superposition of basis states. In many cases, it is useful to represent or "expand" the system's state in the basis of eigenstates of a specific operator $\hat{O}$, where $\hat{O}$ represents a physical property of interest that we want to study or to measure.

3.3 The state-vector and the Schrödinger equation

In quantum mechanics, the complete representation of the state of a system is given by the state vector $|\psi\rangle$. We can thus finally answer one of the initial questions: "how does quantum mechanics describe/represent systems"? Any system–an electron, an atom, a molecule–is mathematically represented by a state-vector, a vector defined in the system's Hilbert space.[15] What is the difference between the state, generally called $|v\rangle$, of the previous section and the state-vector, generally indicated as $|\psi\rangle$? The state-vector is the only state that represents the complete information of the system. It can be equally represented by different (indeed: infinitely many) superpositions of states, but all these superpositions will be different effective descriptions of the vector associated with the system's state: the state-vector $|\psi\rangle$. That is, given a system S with state-vector $|\psi\rangle$, the state-vector may be represented as superposition, for example, of energy eigenstates:

$$|\psi\rangle = \sum_i b_i |E_i\rangle \tag{9}$$

[14] Eigenstates and observables will be analyzed in more detail in the next section

[15] The number of dimensions of the Hilbert space is not fixed but varies from system to system and from different representations of the same system. If we represent a system in the eigenstates of an operators with a finite discrete spectrum, the dimensions will be finite; if we represent the system in the position-basis, the associated Hilbert space will be infinite-dimensional.

or as a superposition of eigenstates of a generic property A (mathematically represented by the operator $\hat{A}$):

$$|\psi\rangle = \sum_i c_i \, |A_i\rangle \tag{10}$$

Equations (9) and (10) will be just different mathematical representations of the same physical state $||\psi\rangle$, the state-vector associated to the system S.
But how do we know how to write the state-vector for a given system? The state-vector is the solution of the Schrödinger equation (3), which we can now put in the correct vector form:

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \hat{H}| \, |\psi\rangle \tag{11}$$

where $i = \sqrt{-1}$ is the imaginary unit, $\hbar = \frac{h}{2\pi}$ (which reads "h-bar") is the reduced Planck constant, $\frac{\partial}{\partial t}$ is the partial derivative over time, $||\psi\rangle$ is the state-vector, $\hat{H}$ is the Hamiltonian operator. The Hamiltonian operator $\hat{H}$ represents the total energy of the system and (as in classical mechanics) is the sum of the kinetic energy and potential energy:

$$\hat{H} = \frac{\hat{P}}{2m} + \hat{V} \tag{12}$$

where $\frac{\hat{P}}{2m}$ is the kinetic energy operator and $\hat{V}$ is the potential energy operator. The Hamiltonian operator is the natural extension in the Hilbert space formalism of the Hamiltonian function in classical mechanics. While in classical mechanics the Hamiltonian H is a function depending on the position and momentum variables of the system $H(x, v)$, in quantum mechanics the Hamiltonian is an operator $\hat{H}$ depending on the position and momentum operators of the system $\hat{x}$ and $\hat{p}$, acting on the state-vector $|\psi\rangle$of the system. Finally, we can write the Schrödinger equation in vector and explicit form as follows:

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \frac{\hat{P}}{2m} \, |\psi\rangle + \hat{V} \, |\psi\rangle \tag{13}$$

The Schrödinger equation defines the state-vector of a system (the state-vector is the solution of (13)) and describes the dynamics of the system: given the Hamiltonian operator $\hat{H}$, it describes how the state-vector evolves in time (this is basically the meaning of the partial derivative on the left-hand side of the equation). In other words: the Schrödinger equation describes the temporal evolution of the state, as in Newtonian mechanics the second law of dynamics $F = ma$ describes how the system will move subject to a force. Note that the "force", or more generally, the causes of motion in the Schrödinger equation are a bit hidden in the Hamiltonian operator. However, when the Hamiltonian operator is written explicitly as in eq. (12) we see that the evolution of the system is due to the kinetic energy and potential energy operators (quite analogously to the classical case in classical Hamiltonian mechanics).

3.4 The Hamiltonian formalism: an example from classical mechanics

Quantum mechanics is written in the Hamiltonian formalism, i.e. the evolution of the state is tracked from the Hamiltonian operator. This is different from the classical picture that we have in Newtonian dynamics, but is similar to the classical picture that we have in classical Hamiltonian mechanics. Let's see the difference between the two.

3.4.1 Newtonian mechanics

In Newtonian mechanics, a system (mathematically represented by a the variable x) accelerates when it is acted upon by a force $F = -\nabla V$, where V is the potential energy, ∇ (which is called "the gradient") is the vector of derivatives with respect to the three position coordinates x, y and z (in one dimension along the $x-$axis, $\nabla = \frac{\partial}{\partial x}$) and F is the classical force. This is quite a mechanical picture: the classical force acts on the system and "pushes" it to move in a certain way. This picture is captured by Newton's second law of dynamics:

$$-\nabla V = m\ddot{x} \tag{14}$$

Suppose, for example, that we want to describe the motion of an object with initial position h_0, where h is the vertical distance from the Earth surface (motion of free fall). The potential energy is:

$$V = mgh$$

and so the Newtonian force is:

$$F = -\nabla V = -\nabla(mgh) = -mg \tag{15}$$

Applying Newton's second law $F = m\ddot{x}$, we find that the object accelerates as follows:

$$-mg = m\ddot{x} \tag{16}$$

that is, the acceleration of the object in free fall is constant and corresponds to the gravitational acceleration ($\ddot{x} = -g$). The sign of the acceleration is negative since the acceleration points towards the center of the Earth and so in anti-parallel to the standard direction of the vertical axis $\vec{z}$. The trajectory of a system in case of uniform accelerated motion is:

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \tag{17}$$

where x_0 and v_0 are, respectively, the initial position and the initial velocity of the object. In our case, $x_0 = h_0$, $v_0 = 0$ (the initial velocity of the object is zero) and $a = -g$. So, the evolution in time of the object in free fall starting from the initial position h_0 is:

$$x(t) = h_0 - \frac{1}{2}gt^2 \tag{18}$$

3.4.2 Hamiltonian mechanics

Let's describe the same example, i.e. an object in free fall with initial position h_0 in the Hamiltonian formulation of classical mechanics. The starting from of Hamiltonian classical mechanics is the Hamiltonian function, which describes the total energy of the system and is composed of the sum of the kinetic E_k and potential energy E_V:

$$H(x, v) = E_k + E_V = \frac{1}{2}mv^2 + V \tag{19}$$

Inserting $V = mgh$, the Hamiltonian becomes:

$$H(x, v) = \frac{1}{2}mv^2 + mgh \tag{20}$$

The motion of the object is computed through the Hamiltonian as follows: the velocity of the system is given by the partial derivative of the Hamiltonian function with respect to the momentum variable ($p = mv$):

$$\dot{x} = \frac{\partial H}{\partial p} \tag{21}$$

and the acceleration by (minus) the partial derivative of the Hamiltonian with respect to the position variable (q):

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{22}$$

where the quantity $\dot{p} = \frac{d}{dt}(mv)$ is the temporal derivative of the momentum and corresponds to the Newtonian force: $\dot{p} = m\dot{v} = ma = m\ddot{x}$. We can compute the acceleration of the object using eq. (22) inserting the Hamiltonian (20):

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial}{\partial q}(\frac{1}{2}mv^2 + mgh) = -mg \tag{23}$$

Recalling that $\dot{p} = m\ddot{x}$, we eventually obtain the same result of Newtonian mechanics:

$$m\ddot{x} = -mg$$

The object with initial position h_0, where h_0 is the vertical distance from the ground, has a uniformly accelerated motion with acceleration $x \doteq -g$ and so the trajectory for an object in free fall computed through Hamilton's equations is:

$$x(t) = h_0 - \frac{1}{2}gt^2 \tag{24}$$

which is the same as the Newtonian trajectory (18), as expected.

3.5 Observables

In quantum mechanics, we call *observable* any measurable physical property that can be assigned to a system. Physical quantities such as energy, momentum ($p = mv$), angular momentum are therefore observables. Any observable is mathematically represented by a *Hermitian operator* $\hat{O}$ acting on the system's Hilbert space. so, first of all:

What is an operator? An operator can be thought of as a mathematical operation which can performed on a vector or a function. In the case of an operator acting on a vector, this will be represented by a matrix. In the case of an operator acting on the wave function, this will be a general mathematical operation such as a derivative or an integral.

Operators are therefore mathematical entities that change the original vector or function defined in Hilbert space into a new vector or function always defined in the same Hilbert space. It is thus a way to modify the original state of the system into a new one. We can thus represent a given observable A as a Hermitian operator $\hat{A}$ acting on the system's state $|\alpha\rangle$. In general, this operation will change the original state into a new state:

$$\hat{A}\,|\alpha\rangle = |\beta\rangle \tag{25}$$

Eigenstates and eigenvalues: given a state $|K\rangle$ and an operator $\hat{A}$, we call the state $|K\rangle$ an *eigenstate* of $\hat{A}$ if the action of the operator on the state does not change the original state. That is, the state $|K\rangle$ is an eigenstate of $\hat{A}$ if the following holds:

$$\hat{A}\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{26}$$

The coefficient α is called the *eigenvalue* of the operator $\hat{A}$ associated to the eigenstate α. In quantum mechanics, all possible eigenvalues of any observable are real numbers (this follows from the identification of observables with Hermitian operators, see below). That is: all measurement outcomes in quantum mechanics are real values and thus can be practically measured in suitable experiment in a laboratory.

Why Hermitian operators? Hermitian operators are *linear self-adjoint* operators. The first characteristic permits to apply the operator to superposition of states by acting on each individual component of the superposition. The second means that the operator $\hat{A}$ is equal to its adjoint $\hat{A}^*$: $\hat{A} = \hat{A}^*$. It is not important for our purpose to analyze this mathematical aspect, but this has a consequence that is very important for the formalism and the interpretation of quantum mechanics: when a linear operator is equal to its adjoint (so, it is self-adjoint), it has a real spectrum of eigenvalues, i.e. all physical properties that can be measured in an experiment have real values. Let's expand a bit on these two features below:

1. **Hermitian operators are linear operators:** they can be applied separately to individual components of a superposition of states. For example, given an observable

represented by $\hat{A}$ and a system represented by the state $|K\rangle = c_1 |k_1\rangle + c_2 |k_2\rangle$, we can applying the operator as follows:

$$\hat{A} |K\rangle = \hat{A} (c_1 |k_1\rangle + c_2 |k_2\rangle) = \hat{A} (c_1 |k_1\rangle) + \hat{A} (c_2 |k_2\rangle) = c_1 \hat{A} |k_1\rangle + c_2 \hat{A} |k_2\rangle \quad (27)$$

2. **Hermitian operators have a real spectrum of eigenvalues:** We call *spectrum* the set of possible eigenvalues of an operator. We call *eigenvalue* the coefficient given by an operator when it is applied to an eigenstate. We call *eigenstate* a state that remain unchanged under the action of an operator. These technical aspects will be explained in more detail below, but for the moment here is the important thing: all physical values that can be measured in an experiment (the eigenvalues) of a generic observable $\hat{A}$ are real-valued. This corresponds indeed to the spectrum of the Hermitian operator. Since Hermitian operators have a real spectrum of eigenvalues, the physical values we will measured in the laboratory (corresponding to the observable's eigenvalues) are real- numbers. Here stands the link between the mathematical formalism of quantum mechanics and the physical values we measure in a laboratory: the eigenvalues of the observables are real numbers, and therefore can be measured in a laboratory.

How to represent observables/Hermitian operators in quantum mechanics

In the Dirac formalism, Hermitian operators are mathematically represented by an external product, i.e. a "ket" followed by a "bra": For example, the density operator $\hat{\rho}$ is written as follows:

$$\hat{\rho} = |\psi\rangle \langle\psi| \quad (28)$$

The density operator is a "projection operator": it projects every initial state $|k\rangle$ into a new state $c_k |\psi\rangle$. suppose, for example, that we apply the density operator $\hat{\rho}$ to a generic state $|k\rangle$. The action of the operator on the state can be described as follows:

$$\hat{\rho} |k\rangle = (|\psi\rangle \langle\psi|) |k\rangle = |\psi\rangle \langle\psi|k\rangle = \langle\psi|k\rangle |\psi\rangle = c_k |\psi\rangle \quad (29)$$

where the coefficient c_k is the projection of the state $|k\rangle$ into the state $|\psi\rangle$: $c_k = \langle\psi|k\rangle$. So, the action of the operator $\hat{\rho}$ transforms the initial state $|k\rangle$ into the new state $c_k |\psi\rangle$. As we will see in the chapter dedicated to decoherence, density operators are particularly important in the quantum-to-classical transition.

An operator in quantum mechanics can be generally represented as a matrix. In this case, the action of the operator on the state is mathematically represented by the product [matrix] X [vector]. Ket-states are represented by column-vectors, whereas bra-states are represented by line-vectors. Consider, for example, the Hermitian operator $\hat{A}$ represented by the matrix:

$$\hat{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (30)$$

The action of $\hat{A}$ on a generic state $|k\rangle$, represented by $|k\rangle = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ is:

$$\hat{A}\,|k\rangle \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a_{11}k_1 + a_{12}k_2 \\ a_{21}k_1 + a_{22}k_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = |\beta\rangle \tag{31}$$

as expected, the result of this operation is a new vector $|\beta\rangle$: the action of the operator transformed the initial state $|k\rangle$ into the new vector $|\beta\rangle$: $\hat{A}\,|k\rangle = |\beta\rangle$

Mean value and uncertainty of an observable

We have seen above that an observable in quantum mechanics represent a physical property that can be measured (ideally or practically) by setting up a suitable experiment in a laboratory. However, when we measure a physical property, this is always to be intended as a property that refers to a given system, represented by a state. Suppose, for example, that the operator $\hat{v}$ represents the observable 'velocity". This gives us a sort of definition of the velocity in the formalism, but, from more a practical point of view, 'we will be interested to calculate the velocity of a given system, i.e. we will be interested in the velocity of a molecule, or a particle, or an atom. In quantum mechanics, the translation of "computing the velocity for a system" is "computing how the observable acts on the state representing the system". That is: always keep in mind that measuring the property of a system means having an operator (representing that property) that acts on the state (representing the system).
In this section, we introduce and define the two main characteristics of an observable: the *mean value* and the *uncertainty*.

Mean value: The mean value of an operator $\hat{A}$ on a state $|\psi\rangle$ is the sum of all possible measurement outcomes (eigenvalues) of a measurement of $\hat{A}$ on the state $|\psi\rangle$, where each outcome is multiplied by the probability to obtain that specific outcome in the measurement. And it is mathematically defined as follows:

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle \tag{32}$$

Example: We compute the mean value of the observable $\hat{A}$ on the state $|S\rangle$

First of all, we have to know the possible measurement outcomes of the observable $\hat{A}$ on the state $|S\rangle$. These are given by the eigenvalues of the observable acting on that specific state. Suppose that the the state $|S\rangle$ can be expanded in the basis of the eigenstates $|\alpha_i\rangle$ of the observable $\hat{A}$ as follows:

$$|S\rangle = c_1\,|\alpha_1\rangle + c_2\,|\alpha_2\rangle \tag{33}$$

with

$$\hat{A}\,|\alpha_1\rangle = a_1\,|\alpha_1\rangle \tag{34}$$

$$\hat{A}\,|\alpha_2\rangle = a_2\,|\alpha_2\rangle \tag{35}$$

That is: the states $|\alpha_1\rangle$ and $|\alpha_2\rangle$ are the eigenstates of the observable $\hat{A}$ and a_1 a_2 the corresponding eigenvalues. If the system has initial state $|\alpha_1\rangle$ (or $|\alpha_2\rangle$), a measurement of $\hat{A}$ on the system will give us the result a_1 (or a_2): a_1 and a_2 are therefore the possible measurement outcomes (the eigenvalues) in a measurement of the observable $\hat{A}$ on the state $|S\rangle$. Finally, we suppose that the different eigenstates are orthogonal with each other (this follows from the characteristics of Hermitian operators):

$$\langle\alpha_1|\alpha_2\rangle = 0 \tag{36}$$

We can now compute the mean value of the observable $\hat{A}$ on the state $|S\rangle$ as follows:

$$\langle\hat{A}\rangle_S = \langle S|\hat{A}|S\rangle = \left(c_1^*\,\langle\alpha_1| + c_2^*\,\langle\alpha_2|\right)\hat{A}\left(c_1\,|\alpha_1\rangle + c_2\,|\alpha_2\rangle\right) =$$

$$= \left(c_1^*\,\langle\alpha_1| + c_2^*\,\langle\alpha_2|\right)\left(c_1\hat{A}\,|\alpha_1\rangle + c_2\hat{A}\,|\alpha_2\rangle\right) =$$

$$= \left(c_1^*\,\langle\alpha_1| + c_2^*\,\langle\alpha_2|\right)\left(c_1 a_1\,|\alpha_1\rangle + c_2 a_2\,|\alpha_2\rangle\right) =$$

$$= c_1^* c_1 a_1\,\langle\alpha_1|\alpha_1\rangle + c_1^* c_2\,\langle\alpha_1|\alpha_2\rangle + c_2^* c_1 a_1\,\langle\alpha_2|\alpha_1\rangle + c_2^* c_2 a_2\,\langle\alpha_2|\alpha_2\rangle =$$

$$= |c_1|^2 a_1 + |c_2|^2 a_2$$

In the last passage, we have used the property of complex numbers: $c_j c_j^* = |c_j|^2$ and the property of orthogonality of states $\langle\alpha_i|\alpha_j\rangle = 0$ for the eigenstates $|\alpha_1\rangle$, $|\alpha_2\rangle$ (eq. (36)).

As expected, we have obtained $\langle\hat{A}\rangle_S = |c_1|^2 a_1 + |c_2|^2 a_2$: the mean value of the observable $\hat{A}$ on the state $|S\rangle$ is the sum of all possible eigenvalues (a_1, a_2) weighted with the corresponding probability to obtain each specific eigenvalue in the measurement $(|c_1|^2, |c_2|^2)$.

Apart from the more abstract formalism, we note that the definition of mean value in quantum mechanics is conceptually the same as in classical mechanics. In order to see the analogy, let's make a standard example from classical mechanics: the mean value of a dice. This is computed as in quantum mechanics: ti is the sum of all possible outcomes weighted with the corresponding probability for each outcome to be realized in a throw of the dice. We have six possible outcomes (the eigenvalues in quantum mechanics): $1, 2, 3, 4, 5, 6$. Each outcome has probability: $P_i = \frac{1}{6}$ (with $i = 1, 2, 3, 4, 5, 6$).

Indicating with p_i the probability associated to the outcome o_i, the "mean value" of the dice will be:

$$\langle M\rangle_D = \sum_i p_i o_i = \frac{1}{6}\left(1 + 2 + 3 + 4 + 5 + 6\right) = \frac{21}{6} = 3.5 \tag{37}$$

Another interesting quantity that characterizes an operator is the uncertainty.

Uncertainty $\Delta\hat{A}$: the uncertainty of an operator $\hat{A}$ is indicated with the symbol $\Delta\hat{A}$ and is mathematically defined as follows:

$$\Delta\hat{A} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2} \tag{38}$$

The uncertainty represents the standard deviation of an empirical distribution: given a distribution composed of a certain number of empirical data, the uncertainty describes how much the data are "dispersed" with respect to the mean value of the distribution. For example, if we represent the empirical distribution as a Gaussian, the uncertainty represents how much the Gaussian is peaked or extended in space. A large Gaussian corresponds to an operator with a high uncertainty, on the opposite a Gaussian peaked around a precise value corresponds to an operator having a small uncertainty. In short: the uncertainty sets the distance over which the empirical data are distributed.

Robertson relation (1929). There is an important relation between the joint uncertainty of two different operators. Given two operators $\hat{A}$ and $\hat{B}$, the product of the two uncertainties (sometimes also called the joint uncertainty) is defined by Robertson relation:

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle| \tag{39}$$

where $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is called the *commutator* of the operators $\hat{A}$ and $\hat{B}$.

At this stage, Robertson's relation may look a bit abstract. However, as we will see in the next section, this is a mathematical premise to derive the Heisenberg position-momentum uncertainty relation.

4 Heisenberg uncertainty principle

Heisenberg position-momentum uncertainty principle is not really a principle (a principle in physics is a postulate that is assumed to be true and that cannot be derived within the theory) but a theorem of quantum mechanics, as it can be mathematically derived from other relations within the theory. In particular, it can be derived from Robertson's relation (39) and the commutator of the position and momentum operator, also called the **canonical commutator**:

$$[\hat{x},\hat{p}] = i\hbar \tag{40}$$

The quantum commutator can be seen as a statement about the "compatibility" of the two operators:

1. If the two operators commute, i.e. $[\hat{A},\hat{B}] = 0$, they shares a common basis of eigenstates, i.e. there exists a basis of states $|\alpha_i\rangle$ such that these states are eigenstates of the operator $\hat{A}$ and the operator $\hat{B}$ simultaneously: $\hat{A}|\alpha_i\rangle = a_i|\alpha_i\rangle$ and $\hat{B}|\alpha_i\rangle = a_i|\alpha_i\rangle$

2. If, instead, two operators do not commute, i.e.$[\hat{A}, \hat{B}] \neq 0$, there does not exist a common basis of eigenstates, i.e. there is no basis of states such that these states are eigenstates of the operator $\hat{A}$ and $\hat{B}$.

The physical implication is the following: if there is no common basis of eigenstates, we cannot measure *simultaneously* the observables corresponding to the two (non-commuting) operators. Remember, indeed, that in order to measure an observable A represented by the operator $\hat{A}$, the initial state of the system must be described by an eigenstate or a superposition of eigenstates of the $A-$operator: If there is no common basis of eigenstates between the operators, say, $\hat{A}$ and $\hat{B}$, it means we can prepare the intial state of the system in a superposition of eigenstates of $\hat{A}$ or eigenstates of $\hat{B}$, but not both simultaneously. We start seeing the implication of the canonical commutator for the Heisenberg principle: since the position and momentum operators do not commute, they cannot be measured simultaneously (they cannot be measured at the same time in the same experiment on the same initial state).
The position-momentum uncertainty relation can be derived from Robertson's relation (39), which is a general relation for computing the joint uncertainty of two operators, and applying this relation to the position and momentum operators using the resul of the canonical commutator (40).
We will then procede in three step:

1. we define the position and momentum operators, both in the position-basis.
2. we derive result (40)
3. we apply Robertson relation (39) to the specific case of the operators $\hat{x}$ and $\hat{p}$

Position operator [definition]: The position operator $\hat{x}$ in the position basis is just x. When the position operator acts on the wave function, it just multiplies the wave function for the variable x:

$$\hat{x}\psi(x) = x\psi(x) \tag{41}$$

where x on the right hand side is the eigenvalue of the position operator, i.e. a specific position in the system configuration space. Please, note: a rigorous description of position eigenvalues would need associating the $\hat{x}-$eigenvalues with Dirac's delta functions. This is in fact the standard characterization that is generally found in quantum mechanics textbooks. However, this goes after the scope of this introduction to the formalism, and the link of eigenstates-eigenvalue sketched above is a good approximation to derive our results below.

Momentum operator [definition]: The momentum operator $\hat{p}_x$, represented in the position basis and acting along the coordinate $\vec{x}$ is defined by the following operation:

$$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x} \tag{42}$$

In general, the partial derivative ($\frac{\partial}{\partial x}$) will be substituted by the vector of partial derivative along the $\vec{x}$, $\vec{y}$ and $\vec{z}$ axis (∇). A more general formulation is thus the following:

$$\hat{p} = \frac{\hbar}{i}\nabla \tag{43}$$

The momentum operator applied to the wave function is essentially a derivative with respect to the position variable.

Derivation of the canonical commutator: $[\hat{x}, \hat{p}] = i\hbar$

The starting point is to express the quantum commutator explicitly as the difference of products of the two operators and recalling that an operator is meaningless on its own: it always acts on a state or on a function. In this case, since both operators are expressed in the position basis, the commutator naturally acts on a function, which is generally written in the generic form f (but that implicitly depends on position: $f(x)$). So we have:

$$[\hat{x}, \hat{p}]f = (\hat{x}\hat{p} - \hat{p}\hat{x})\, f = \left(x\left(-i\hbar\frac{\partial}{\partial x}\right)\right) f - \left(\left(-i\hbar\frac{\partial}{\partial x}\right)(x)\right) f = x\left(-i\hbar\frac{\partial f}{\partial x}\right) - \left(-i\hbar\frac{\partial (xf)}{\partial x}\right) =$$

$$= -i\hbar\frac{\partial f}{\partial x}x + i\hbar\left(\frac{\partial x}{\partial x}f + \frac{\partial f}{\partial x}x\right) = i\hbar\frac{\partial f}{\partial x}x + i\hbar f + i\hbar\frac{\partial f}{\partial x}x = i\hbar f$$

From the derivation above we have the equality $[\hat{x}, \hat{p}]f = i\hbar f$ and so the value of the canonical commutator

$$[\hat{x}, \hat{p}] = i\hbar$$

4.1 The position-momentum uncertainty relation

The Heisenberg uncertainty principle for the position and momentum operators, i.e. the position-momentum uncertainty relation, follows from Robertson relation (39) which describes the joint uncertainty of two generic observables, applied to the position and momentum operators. inserting (40) into (39) we have:

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{1}{2}|\,\langle[\hat{x}, \hat{p}]\rangle\,| = \frac{1}{2}|\,\langle i\hbar\rangle\,| = \frac{1}{2}|i\hbar| = \frac{\hbar}{2} \tag{44}$$

We have thus derived the position-momentum uncertainty relation:

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{\hbar}{2} \tag{45}$$

From the mathematical point of view, the Heisenberg uncertainty principle follows from the general relations about the product of the uncertainties of two operators (Robertson relation) applied to the specific case of the position and momentum operators. The principle is standardly expressed as follows:

> *It is impossible to measure the position and velocity of a quantum system simultaneously and with infinite precision (no uncertainty).*

But what is the meaning of the uncertainty principle? In the literature, it is common to find two different interpretations of this principle:

1. The ontological interpretation
2. The epistemic interpretation

Ontological interpretation: The uncertainty principle refers to individual systems and thus have an ontological status concerning the description of systems. If we cannot measure simultaneously and exactly the position and momentum of a system, it means the system itself *does not have* an exact position and an exact velocity. It follows a quantum system does not move along a trajectory in space. From a mathematical point of view, exact position and momentum values are needed indeed to represent a continuous trajectory. The uncertainty principle, according to the ontological interpretation, sets a limitation in the ontology of systems: systems do not have a well-defined position and velocity at the same time, and therefore do not move along continuous trajectories. Quantum mechanics tells us that the dynamics of quantum systems is therefore qualitatively different from that one of classical systems. The very concept of trajectory cannot be applied to quantum systems. This kind of interpretation fits particularly well in a positivistic framework, as it implicitly assumes the following statement: *if* we cannot measure exact position and velocity at the same time, *then* the system does not have an exact position and velocity. In the positivist view, all that exists (in the ontological sense) are the things we can measure. If something is not measurable according to the theory (so, not for practical difficulties, but in principle), then it does not exist according to the theory, i.e. it is not in the ontology. This interpretation is typically adopted in the standard interpretation of quantum mechanics and, therefore, presented as a fundamental feature of quantum systems in most quantum mechanics textbooks: see, for example, Sakurai (1994).

Epistemic interpretation The uncertainty principle refers to an ensemble of systems, i.e. it cannot be applied to individual systems but only to a collection of many systems with the same initial state. This interpretation gives priority to the way the relation can be really implemented in a laboratory. Suppose that we are in a laboratory and we want to measure the position and the momentum of a system. First of all, we cannot measure them simultaneously not because of the uncertainty principle, but because the two operators do not commute (the canonical commutator is different from zero). We know from the basic formalism of quantum mechanics that, if two operators do not commute, these two operators do not share a common basis of eigenstates. And, for this reason, we cannot perform a measurement of the position and momentum observables simultaneously. Second, we decide then to measure the position of the system. The initial state of the system must be prepared in a superposition of position eigenstates, and the eigenvalue will be one of the eigenstates with probability given by the Born's rule. The eigenvalue will be a precise, definite point (fo r example, a dot on the screen). There is no uncertainty on this eigenvalue

if not the inevitable uncertainty given to the sensitivity of the measurement device. That is: when we perform a single (position or momentum) measurement on a single system, the uncertainty relation is meaningless. In order to see the uncertainty relation being realized, we must perform many measurements of the position and many measurement of the momentum. in this way, we will have as a result two empirical distributions: an empirical distribution of the position eigenvalues and an empirical distribution of the momentum eigenvalues. From these distributions, we can compute the mean value of the position and the momentum operator. From the mean value, we can eventually compute the uncertainty/standard deviation of the position and momentum operators as indicated in (38). According to this interpretation, the inequality of the uncertainty principle sets a lower bound on the precision through which we can measure the position and momentum in a series of measurements of identically prepared systems, but it does not say anything on the nature of individual systems. Bohm's theory is an example of quantum theory in which individual systems have exact position and velocity and yet statistically they obey the constraints of the uncertainty principle. This interpretation has been originally proposed by Margenau (1963) and more recently defended by Ballentine (1970), Ballentine (1998), Bowman (2008), Romano (2021).

5 The theory of measurement

The description of quantum mechanics we have seen so far goes as follows:

1. Given a system (typically: an electron, an atom, a molecule), the state of the system is represented by a vector, the state-vector $|\psi\rangle$ in an abstract vector space called the Hilbert space.

2. As a vector can be represented as a superposition (linear combination) of other vectors (the basis vectors), the state-vector can be represented as a superposition of other states, the basis states. Therefore, a typical quantum state will be generally written as: $|\psi\rangle = c_1 |\alpha_1\rangle + c_2 |\alpha_2\rangle + \ldots + c_N |\alpha_N\rangle = \sum_{i=1}^{N} |\alpha_i\rangle$

3. The states $|\alpha_i\rangle$ are generally chosen to be eigenstates of a given observable $\hat{A}$:

$$\hat{A} |\alpha_i\rangle = a_i |\alpha_i\rangle$$

   where the observable $\hat{A}$ represents mathematically a specific physical property of the system that we want to measure or to study (e.g., position, momentum, energy, angular momentum).

4. From the last point: if the state-vector is an eigenstate of the observable $\hat{A}$[16]:

$$\hat{A} |\psi\rangle = a_\psi |\psi\rangle$$

[16] For simplicity, I just describe $\hat{A}$ as an observable. To be precise, however, $\hat{A}$ is an Hermitian operator that mathematically represents a given observable, i.e. a given measurable physical property.

, then the system has a definite value with respect to the observable $\hat{A}$. The definite value a_ψ is a real-valued number and is called the eigenvalue For example, if the observable $\hat{A}$ represents the momentum, then the state represents a system with a definite momentum (i.e. a definite velocity) and the value of the momentum is a_ψ. If we perform a measurement of the momentum, the measurement outcome will be with certainty the value a_ψ.

5. If the state of the system is a superposition of eigenstates of a given observable (this is indeed the most general and typical case):

$$|\psi\rangle = \sum_{i=1}^{N} |\alpha_i\rangle$$

then the state does not have a definite value with respect to that observable. However, in a measurement of the observable, the state will collapse in one of the eigenstates of the superposition:

$$|\psi\rangle = \sum_{i=1}^{N} c_i |\alpha_i\rangle \xrightarrow{H_{int}} |\alpha_k\rangle \tag{46}$$

This is a the postulate of the **collapse of the wave function** and it applies only in measurement interactions. The probability that the initial state will collapse in a specific eigenstate of the superposition, say $|alpha_k\rangle$ is given by the absolute square of the expansion coefficient $|c_k|^2$:

$$P\left(|\psi\rangle \to |\alpha_k\rangle\right) = |c_k|^2 \tag{47}$$

In the philosophical literature, this is generally called the **Born rule**.

The **theory of measurement** is the sum of the collapse postulate and the Born rule and may be resumed as follows:

> Given a system S with state $|\psi\rangle$, where $|\psi\rangle$ a superposition of eigenstates of a certain operator $\hat{A}$: $|\psi\rangle = \sum_i c_i |\alpha_i\rangle$, if we perform a measurement of the observable represented by the operator $\hat{A}$ the initial state $|\psi\rangle$ will collapse in one of the eigenstates of the superposition $|\alpha_i\rangle$ with probability given by the corresponding coefficient $P\left(|\psi\rangle \to |\alpha_i\rangle\right) = |c_i|^2$.

Note that the Born rule presented in the theory of measurement is essentially the Born's statistical interpretation of the wave function applied to the case of vectors and discrete representations (superposition of discrete eigenstates). The standard view of quantum mechanics therefore predicts two different dynamical processes:

- **The Schrödinger equation**, when the system is isolated. This is a deterministic, linear and reversible dynamics. It describes a continuous evolution of the state, very similarly to the dynamical evolution of a wave but in a more abstract high dimensional configuration space.

- **Wave function collapse**: when the system interacts with a measurement apparatus, the state vector or the wave function collapse in one of the eigenstates of the measure observable. This kind of dynamics is the opposite of the Schrödiger evolution: the collapse process is indeterministic (intrinsically, ontologically probabilistic), non-linear and irreversible. The reason why is ontologically probabilistic is the following: according to standard quantum mechanics, the state-vector or the wave function provides the complete information on the system, but still, even if we have maximum knowledge on the system, we cannot predict, if not probabilistically, in which specific eigenstate the initial superposed state will collapse during the measurement interaction.

5.1 The measurement problem

In the previous section we have seen that, in a measurement, the initial state of the system, which is typically a superposition of eigenstates of the measured observable, will collapse onto one of the eigenstates of the superposition with probability given by the Born rule. However, so far we have not included the state of the apparatus in this representation. The measurement apparatus is a physical system as the quantum system that interacts with it. In a more accurate description of the measurement process, we should include the apparatus in the general picture and describe the measurement interaction as an interaction between the system and the apparatus. The measurement apparatus, i.e. the macroscopic device through which we perform a measurement of a given observable on the system, is generally indicated with $|A\rangle$. Consider a system with state-vector $|\psi\rangle$, which is a superposition of eigenstates of the observable measured by the apparatus. That is: the system is described by the following superposition:

$$|\psi\rangle = \sum_i c_i |\psi_i\rangle$$

with

$$\hat{A} |\psi_i\rangle = a_i |\psi_i\rangle$$

where the apparatus $|A\rangle$ is designed to measure the observable A mathematically represented by the operator $\hat{A}$. The interaction between the system and the apparatus will be described as follows:

$$|\psi\rangle |A\rangle = \left(\sum_i c_i |\psi_i\rangle \right) |A\rangle \xrightarrow{\hat{H}_{int}} \sum_i c_i |\psi_i\rangle |A_i\rangle \tag{48}$$

Each apparatus state $|A_i\rangle$ can be thought as a different macroscopic configuration of the measurement device, for example (thinking of the apparatus as a pointer) as

a macroscopic pointer pointing in a definite direction. But then, the final state of the measurement process described by (48) is a superposition of different well-defined macroscopic states, a physical situation that is never observed in nature. In particular, the superposition (48) represents a direct correlation between system states and apparatus states: each relative-state of the system $|\psi_i\rangle$ is correlated to a definite relative-state of the apparatus $|A_i\rangle$. A state of this kind is called an entangled state: note that, when the entangled state is formed, we cannot assign anymore a state-vector to the system and the apparatus separately. We can assign only a state-vector only to the composite system-apparatus entangled state:

$$\sum_i c_i \, |\psi_i\rangle \, |A_i\rangle = |\Psi\rangle = |\psi, A\rangle \tag{49}$$

It is important to remark that, up to this point, the collapse of the wave function has not taken place yet. What we have described so far is the interaction between the system and the apparatus though the usual Schrödinger evolution. The symbol $\hat{H}_{int}$ represents indeed the interaction Hamiltonian of the Schrödinger equation, through which the two systems are coupled together. Up to this moment, a real measurement has not been performed according to standard quantum mechanics. This is realized in fact when the collapse of the wave function takes place: once the system and the apparatus have interacted with each other and formed the entangled state (49), the composite system-apparatus state will collapse in one of the components of the superposition $\sum_i c_i \psi_i A_i$, each component representing a certain eigenstate of the measured observable and a definite macroscopic configuration of the apparatus measuring the eigenvalue associated tt that eigenstate:

$$|\Psi\rangle = \sum_i c_i \, |\psi_i\rangle \, |A_i\rangle \xrightarrow{\hat{A}} |\psi_k\rangle \, |A_k\rangle \quad \textit{with} \quad \textit{probability} \quad P = |c_k|^2 \tag{50}$$

The measurement problem is the problem to make sense of the entangled system-apparatus state (49), which describes a superposition of different well-defined macroscopic states of the apparatus. In the standard view of quantum mechanics, this problem is easily solved by the collapse of the wave function. Once the wave function collapse, there is only one state for the system and one configuration for the apparatus. From a practical point of view, the reason why we never observe such kind of superpositions is pretty clear: the observation (in general, a measurement) implies the collapse of the wave function, which in turn washes away the superposition of the apparatus states. However, the philosophy community is not particularly satisfied by that answer: the collapse postulate does not look like a law of nature (as the Schrödinger equation) or a physical dynamical process of any sort, rather it looks like as an *ad hoc* practical strategy to eliminate the problem and unify the connection between the formalism of quantum mechanics and the classical world. In the next section, we will analyze the measurement problem in more detail.

5.2 Some misconceptions about the measurement problem

In the philosophical literature, the measurement problem is generally presented with the following statement (or minimal variations of this statement):

> When we perform a measurement on the system, the latter gets entangled with the apparatus state and the Schrödinger evolution (after the interaction has happened) describes a coherent superposition of macroscopically well-defined but incompatible states, that is, the different macroscopic states of the measurement device registering multiple definite measurement outcomes. But this is in contradiction with the empirical result of the measurement, which always corresponds to one particular measurement outcome.

This statement however is not correct if not implemented with other considerations. In standard quantum mechanics, a measurement is first and foremost characterized by the collapse of the wave function. It is true that the Schrödinger evolution *at some point* describes a superposition of apparatus macroscopic different states. However, in standard quantum mechanics the wave function is not a real entity and thus it does not represent a material field of any sort composing the objects. That is: a superposition of different macroscopic positions of the apparatus does not correspond to a material or ontological superposition. As we have seen in section 1 and section 2, the wave function in the standard view is a mathematical entity which gives the probability distribution of the measurement outcomes. This brings a lack of description (what is the ontology of the physical world when we do not measure it?) but it does not lead to a material superposition of different apparatus states. In particular, we have seen that the probability of the measurement outcomes is assigned according to Born's statistical interpretation, given by the absolute square of the expansion coefficients.

This point is important to understand that according to the physics community there is simply no measurement problem or, in any case, it is generally regarded as a philosophical problem that should not have any impact on the physics of quantum mechanics. This is the point where the physics and the philosophy community diverge: the former does not consider the measurement problem as a serious problem of the theory and so adopts unproblematically the standard view of quantum mechanics. After all, it is the simplest theory that predicts the result of any experiment in the quantum regime and it is more flexible than Bohm's theory and GRW theory.[17] With "more flexible", I mean the three following points:

1. The mathematics and physics of Bohm's theory and GRW theory is more complicated than standard quantum mechanics. Bohm's theory adds to the standard formalism new variables (the position variables, which describe the exact position of

[17] It is excluded from this problem the Everett theory or Many Worlds Interpretation of quantum mechanics, which is not a different theory (as Bohm's theory and GRW theory) but a metaphysiocal re-interpretation of the standard formalism.

the particles) and a new dynamical equation (the dynamical evolution of the particle position). The GRW theory adds a non-linear term in the Schrödinger equation which leads to a spontaneous (dynamically driven) collapse of the wave function at random times and random points (but always absolute squared distributed).

2. Standard quantum mechanics has been extended to the regime of special relativity, i.e. we have a well-defined (standard) relativistic quantum mechanics, while the same relativistic extension for Bohm's theory and GRW theory is more difficult to achieve for either conceptual and technical reasons.[18]

3. Based on the relativistic extension of quantum mechanics, we have in the standard framework a generalization of quantum mechanics called *Quantum Field Theory* (QFT). QFT is the mathematical basis for the physics of elementary particles, whose principal sectors are Quantum Electrodynamics (QED) and Quantum Cromodynamics (QCD). The former is the theory that explains the interaction between light (photons) and matter (fermions), the latter describes the physics of quarks inside of the atomic nuclei. QED and QCD together describes the physics of elementary particles, i.e. the *Standard Model* of particle physics.

An influential presentation of the measurement problem has been given by Maudlin (1995). In this paper, Maudlin presents the measurement problem as a contradiction between the following three statements:

1. The wave function is the complete representation of the state of the system
2. The Schrödinger equation describes the evolution of all quantum systems
3. The measurement interaction has one specific outcome

Maudlin famously presents the proposals of non-standard interpretation of quantum mechanics (Bohm's theory, GRW theory, MWI) as specific modifications of one of these three original statements of quantum mechanics. In particular:

- Bohm's theory drops the first assumption. The complete representation of an N-particle system in Bohm's theory is supplemented by the actual configuration of N particle positions ($Q = q_1, q_2, \ldots, q_N$) and so the state of a system is represented by the couple: (Q, ψ), where $Q(q_1, \ldots, q_N)$ is the particle configuration and $\psi(x_1, \ldots, x_N)$ is the wave function.
- GRW theory drops the second assumption. In GRW theory, the Schrödinger equation is modified introducing a non-linear term, which is responsible for the spontaneous collapse of the wave function.

[18] This is a tricky and subtle issue, but a precise analysis of this topic lays outside the scope of this presentation. For the moment, it suffices to say that there are in the community there is no consensus on this topic. For example, some influential philosopher of physics (e.g. Maudlin) claims that the collapse of the wave function is clearly a non-local phenomenon, and so standard quantum mechanics is not fundamentally relativistic. On the other hand, relativistic versions have been proposed in the framework of Bohm's theory and GRW theory.

- MWI drops the third assumption. In the Everett theory or MWI, the measurement process does not have a unique outcome: all components of the system-apparatus superposition are realized, each in a different branch of the many worlds ontology.

Also in this case, we see that Maudlin's assumptions are too strong for the standard interpretation. As we noted before, the first assumption is mathematically correct but it does not imply that the system-apparatus entangled state describes a physical superposition of different macroscopic states. The second assumption is also too strong: in the measurement process, soon after the interaction between the system and the apparatus the Schrödinger equation is substituted by the wave function collapse. We may certainly criticize this dichotomy (claiming e.g. that the collapse does not look like a dynamical law or that the theory does not specify when the collapse occurs), but this is a different to say (or to give the impression) that quantum mechanics relies on contradictory statements. In the standard view, the collapse makes it possible to obtain a unique measurement result from the initial superposition, though the specific result is intrinsically probabilistic. To resume: we cannot say that quantum mechanics is in some way self-contradictory or that relies on self-contradictory assumptions. What we can say is that quantum mechanics correctly predicts the empirical distribution of measurement outcomes thanks to the collapse postulate and the Born's rule, but it fails to provide an objective description of systems before and independently from the measurement. This point will be expanded in the next section.

5.3 Three sub-problems for the measurement problem

Leaving aside the misconceptions presented in the last section, the measurement problem can be framed in terms of the following three questions, each individuating a different "sub-problem" for the measurement problem.[19]

1. **What is a measurement?** According to the collapse postulate, the wave function collapses when a measurement is performed, but it is not specified within the theory what counts as a measurement and therefore when a measurement is performed. Usually, a measurement interaction is defined as an interaction between a microscopic system and a *macroscopic apparatus*, but, we may ask, "how much macroscopic" has to be an apparatus to become a measurement apparatus? The problem is just shifted, as there is no precise boundary between microscopic and macroscopic systems.

[19] The analysis presented here will mainly follow the analysis of the measurement problem made by Norsen (2017). However, I do not always agree with Norsen's analysis of the problem. For example, in section 3.1 Norsen assumes that measurement apparata are classical systems. This was historically proposed by the Copenhagen interpretation but quickly dismantled. In fact, measurement apparata are treated quantum-mechanically already in von Neumann's seminal book: *Mathematical Foundations of Quantum Mechanics* (1932), which provides the standard description for the system-apparatus measurement interaction. Also, in many points of the analysis Norsen seems to assume that the wave function in quantum mechanics describes systems in realistic terms, which it does not.

2. **Why a measurement?** Even assuming that we have a good answer to the first question, i.e., that we have a good definition of what counts and what does not count as a measurement apparatus, still we face another problem: why are measurement interactions described as special interactions? After all, the interaction between a microscopic system and a macroscopic system, or equivalently between a quantum system and a measurement apparatus, is a normal interaction between two physical systems. Then why does this kind of interaction require a special law (the wave function collapse) that does not apply to all other physical interactions? A direct answer seems to be: the wave function collapse is exactly what we need to describe the measurement results. On this regard, it seems a pragmatic tool rather than a fundamental physical law. In fact, many discussions on the measurement problem just take this point for granted and try to propose a solution to the problem without the collapse postulate. The most important solutions to measurement problem in this direction are the non-standard interpretations of quantum mechanics:[20] Bohm's theory, GRW theory and Everett theory/MWI.

Solutions to the measurement problem Sometimes the philosophical literature iterates the impression that non-standard theories such as Bohm's theory and GRW theory have been formulated to solve the measurement problem. However, this is not correct. For example: Bohm's theory was formulated to provide a realist interpretation of quantum mechanics, independently form the measurement problem, while GRW theory presented an elegant unification for the dynamics of microscopic and macroscopic systems. This is, for example, the abstract of Bohm (1952). the seminal paper in which Bohm proposes his new interpretation of quantum theory in terms of "hidden" variables:

> The usual interpretation of the quantum theory is self-consistent, but it involves an assumption that cannot be tested experimentally, *viz.*, that the most complete possible specification of an individual system is in terms of a wave function that determines only probable results of actual measurement processes. The only way of investigating the truth of this assumption is by trying to find some other interpretation of the quantum theory in terms of at present "hidden" variables, which in principle determine the precise behavior of an individual system, but which are in practice averaged over in measurements of the types that can now be carried out. In this paper and in a subsequent paper, an interpretation of the quantum theory in terms of just such "hidden" variables is suggested. It is shown that as long as the mathematical theory retains its present general form, this suggested interpretation leads to precisely the same results for all physical processes as does the usual interpretation. Nevertheless, the suggested interpretation provides a broader conceptual framework than the usual interpretation, because it makes possible a precise and continuous description of all processes, even at the quantum level. This broader

[20] For this reason, they are sometimes also called *observer-independent interpretations*

> conceptual framework allows more general mathematical formulations of the theory than those allowed by the usual interpretation. Now, the usual mathematical formulation seems to lead to insoluble difficulties when it is extrapolated into the domain of distances of the order of 10^{-13} cm or less. It is therefore entirely possible that the interpretation suggested here may be needed for the resolution of these difficulties. In any case, the mere possibility of such an interpretation proves that it is not necessary for us to give up a precise, rational, and objective description of individual systems at a quantum level of accuracy.

There are many interesting passages in this abstract. We note, for example, that Bohm acknowledges right at the beginning that quantum mechanics is self consistent. It claims however that the completeness of the wave function is an assumption that cannot be experimentally tested and thus can be challenged on the theoretical ground (in this case, assuming the opposite, i.e. that the wave function does not represent completely the system). Concerning the motivation for this new theory, Bohm claims that it *makes possible a precise and continuous description of all processes*. With the term "all processes", Bohm means the most important textbook examples, such as the particle in a box and quantum tunneling. Remarkably enough, the measurement process will be analyzed only in a subsequent paper (part II). A second motivation is related to *insoluble difficulties when [quantum mechanics] is extrapolated into the domain of distances of the order of* 10^{-13} *cm or less*. On that period, indeed, QFT was not yet well-defined. Bohm's idea was that a more precise formulation of quantum mechanics in the non-relativistic regime would have helped to find the required generalization in the atomic domain. Therefore, we see that the motivations provided fall under two boxes: (i) the possibility of a more accurate description for the physical processes of individual systems and (ii) the possibility of a generalization/extension of the theory to the yet unknown domain of elementary particles. The solution of the measurement process is not given as a motivation for the formulation of the new theory, even though a precise description of quantum processes will naturally lead to the solution of that problem.

3. **What's before measurement?** Finally, a third, subtle problem. Before the collapse takes place, quantum mechanics describes a superposition of different macroscopic states. What does this superposition represent? If this superposition does not have an ontological meaning but it only represents a probability distribution of the possible measurement outcomes, then we have a serious problem of description. It seems, indeed, that quantum theory is unable to provide an objective description of systems unless those systems are measured. But what we do expect from a fundamental physical theory is instead an objective description of systems, independently on whether they are measured or not. Within this positivist attitude, and insofar this positivist attitude is consistently maintained, quantum mechanics is unable to answer ontological questions about microscopic systems. This is where Bohm's, GRW and Everett theories enter into the picture: all these theories provide a description of nature independently from measurement interactions.

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